Thus for v_0 ctg $\beta = a$ the maximum force derived by the theory of incompressible fluid exceeds by 62 % that calculated for a compressible fluid.

The author thanks E. I. Grigoliuk and S. S. Grigorian for discussing this problem.

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Translated by J. J. D.

UDC 532.517.4

ASYMMETRIC MECHANICS OF TURBULENT FLOWS. ENERGY AND ENTROPY

PMM Vol. 37, №1, 1973, pp. 94-105
V. N. NIKOLAEVSKII
(Moscow)
(Received May 22, 1972)

Averaged equations of motion of a turbulized fluid in the presence of a preferred orientation of turbulent vortices were constructed in [1]. By taking account of an additional kinematic variable, the angular velocity of vortex self-rotation, the system of equations in [1] differs from the earlier theory of Mattioli [2].

The equations from [1] are supplemented herein by a turbulent energy balance equation in which the work of the moment stresses and the antisymmetric component of the Reynolds stress tensor is taken into account. It is shown that the inner energy determined by turbulization of the fluid depends on the root-mean-square values of the translational pulsation velocities and the anglular vortex velocities. The entropy and "temperature" of turbulization are introduced; the

entropy production equations are formed. The use of the Onsager formalism of the thermodynamics of irreversible processes is discussed. The stationary state of the system, characterized by an influx of negative entropy (this latter is typical for biological systems [3, 4]) and a constant rate of entropy production, is considered.

1. Mass, momentum, and moment of momentum balance. The balance equations of the mass, momentum, and moment of momentum of a nonpolar fluid can be represented as [1]

$$\frac{\partial}{\partial t}\langle\rho\rangle + \frac{\partial}{\partial X_j}\langle\rho u_j\rangle_j = 0$$

$$\frac{\partial}{\partial t}\langle\rho u_i\rangle + \frac{\partial}{\partial X_j}\langle\rho u_i u_j\rangle_j = \frac{\partial\langle t_{ij}\rangle_j}{\partial X_j} + \langle F_i\rangle \tag{1.1}$$

$$\frac{\partial}{\partial t}\langle\varepsilon_{ilk}\rho u_l\xi_k\rangle + \frac{\partial}{\partial X_j}\langle\varepsilon_{ilk}\rho u_l\xi_k u_j\rangle_j + \langle\varepsilon_{ilk}\rho u_l u_k\rangle_k =$$

$$\frac{\partial}{\partial X_j}\langle\varepsilon_{ilk}\xi_l t_{kj}\rangle_j + \langle\varepsilon_{ilk}t_{lk}\rangle_k + \langle\varepsilon_{ilk}\xi_l F_k\rangle$$

Here ρ is the fluid density, U_j the velocity, F_j the volume force, ε_{ilk} the alternating Levi-Civita tensor, $\langle\ \ \rangle$ the symbol for averaging with respect to a volume element in the space $V = \Delta X_1\Delta X_2\Delta X_3$, X_j coordinates of the center of gravity of the volume V, $\xi_j = x_j - X_j$ is the coordinate of a point within the volume V relative to the center of gravity X_j; $\langle\ \ \rangle_j$ is the symbol for the average over of the face of the volume V to which the X_j axis is normal. Let us examine the case when the fluid flow in the volume $dV = d\xi_1 d\xi_2 d\xi_3$ satisfies the Navier-Stokes equations, i.e.

$$t_{ij} \equiv t_{ji} = \left(-p + \frac{2}{3}\rho\nu\frac{\partial u_k}{\partial\xi_k}\right)\delta_{ij} + \rho\nu\left(\frac{\partial u_j}{\partial\xi_i} + \frac{\partial u_i}{\partial\xi_j}\right), \quad dV \ll V$$

where p is the pressure, ν the kinematic viscosity, and δ_{ij} the unit tensor. The velocity field in the volume $V \sim \Delta^3$ is represented as [1]

$$u_i(x_k, t) = U_i(X_k, t) + \frac{\partial U_i}{\partial X_k}(x_k - X_k) + v_i(x_k, t) \tag{1.2}$$

where U_i is the mean mass velocity of the fluid, v_i is an irregular component (pulsation) of the velocity. In the scale d ($\Delta \gg d$) the quantity v_i is also representable as the first two members of the Taylor series

$$v_i(\xi_k) = w_i(\zeta_k) + (\partial w_i / d\zeta_k)(\xi_k - \zeta_k)$$

i.e. the following

$$u_i(x_k, t) = U_i(X_k, t) + \frac{\partial U_i}{\partial X_k}\xi_k + w_i(\zeta_k, t) + \frac{\partial w_i}{\partial\zeta_k}(\xi_k - \zeta_k) \tag{1.3}$$

is valid in the volumes $\Delta V \sim d^3$ instead of the representation (1.2). Here ζ_k is the coordinate of the center of gravity of ΔV. Averaging the field (1.3) with respect to the volume ΔV yields

$$\bar{u}_i(\zeta_k, t) = U_i(X_k, t) + \frac{\partial U_i}{\partial X_k}\zeta_k + w_i(\zeta_k, t)$$

The elementary moment of momentum m_i can be represented correspondingly as

$$m_i = \varepsilon_{ijk}\rho u_j\xi_k = \varepsilon_{ijk}\rho\left(U_j + \frac{\partial U_j}{\partial X_m}\xi_m + w_j + \frac{\partial w_j}{\partial\zeta_m}(\xi_m - \zeta_m)\right)\xi_k$$

Let us take the average of m_i over the volume ΔV. We obtain

$$\overline{m}_i = \varepsilon_{ijk}\left(\bar{\rho}U_j\zeta_k + \frac{\partial U_j}{\partial X_m}\bar{\rho}\zeta_m\zeta_k + \bar{\rho}w_j\zeta_k\right) + \varepsilon_{ijk}\left(\frac{\partial U_j}{\partial X_m} + \frac{\partial w_j}{\partial \zeta_m}\right) i_{mk}$$

$$i_{mk} = \frac{1}{\Delta V}\int\limits_{\Delta V} \rho\,(\xi_m - \zeta_m)(\xi_k - \zeta_k)\,d\xi_1\,d\xi_2\,d\xi_3$$

where i_{mk} is the specific moment of inertia [1] of the fluid in the volume ΔV. Then, taking the average $\overline{m}_i$ over all the volumes ΔV contained in V, we obtain

$$\langle m_i\rangle = \varepsilon_{ijk}\frac{\partial U_j}{\partial X_m} I_{mk} + M_i, \qquad M_i = \varepsilon_{ijk}\left\langle\left(\frac{\partial U_j}{\partial X_m} + \frac{\partial w_j}{\partial \zeta_m}\right) i_{mk}\right\rangle \quad (1.4)$$

$$I_{mk} = \frac{1}{V}\int\limits_V \bar{\rho}\zeta_k\zeta_m\,d\zeta_1\,d\zeta_2\,d\zeta_3, \quad \varepsilon_{ijk}\frac{1}{V}\int\limits_V \bar{\rho}w_j\zeta_k\,d\zeta_1\,d\zeta_2\,d\zeta_3 = 0$$

Here I_{mk} is the specific moment of inertia [5] of the fluid in the volume V and the condition imposed on the field w_j corresponds to the simplifying assumption that only turbulent vortices of scale d are moment of momentum carriers. As in [1], let us neglect the first summand in the first item of (1.4) in the case of high turbulization, i.e. let us set $\langle m_i\rangle \approx M_i$. If the volumes ΔV are symmetrical, then $i_{mk} = \frac{1}{2}\,i\delta_{mk}$ and also

$$M_i = \langle i\Phi_i\rangle, \qquad \Phi_i = \frac{1}{2}\varepsilon_{ijk}\left\langle\frac{\partial U_j}{\partial X_k} + \frac{\partial w_j}{\partial \zeta_k}\right\rangle$$

where the mean field of natural angular velocities ω_i can be introduced [1] such that

$$M_i = J(\Omega_i + \omega_i), \qquad J\omega_i = \langle i^*\Phi_i^*\rangle$$

$$J = \langle i\rangle, \quad \Omega_i = \langle\Phi_i\rangle = \frac{1}{2}\varepsilon_{ijk}\frac{\partial U_j}{\partial X_k}, \quad J\omega_i^* = i^*\Phi_i^* - \langle i^*\Phi_i^*\rangle$$

where the asterisk denotes the pulsation. As regards the pulsation M_i^* of the moment of momentum, in conformity with the above it is then defined as follows:

$$M_i + M_i^* = \varepsilon_{ijk}\,\rho u_j\xi_k \quad (1.5)$$

The momentum flux generated by turbulence is represented, as is known, in the form

$$\langle\rho u_i u_j\rangle_j \approx \langle\rho\rangle\, U_i U_j - R_{ij} \quad (1.6)$$

where R_{ij} is the Reynolds stress. According to [1], we have

$$R_{ij} = -\langle\rho v_i v_j\rangle_j$$

i.e. the tensor R_{ij} generally contains antisymmetric components. As is customary in hydromechanics the momentum flux associated with the mean velocity gradient in the scale Δ is neglected in (1.6). By using the representation (1.3), it can also be shown that R_{ij} consists of the two components

$$R_{ij} = -\langle\rho w_i w_j\rangle_j - \left\langle\left(\frac{\partial U_i}{\partial X_k} + \frac{\partial w_i}{\partial \zeta_k}\right)\left(\frac{\partial U_j}{\partial X_m} + \frac{\partial w_j}{\partial \zeta_m}\right) i_{km}\right\rangle_j$$

The former corresponds to the mean translational pulsations of the fluid in the volume ΔV, and the latter is due to its rotations around the centers of mass ΔV. Let us represent the moment of momentum flux [1] in conformity with the relationship (1.5) as (μ_{ij} are the moment stresses)

$$\varepsilon_{ilk} \langle \rho u_l \xi_k u_j \rangle_j \approx M_i U_j - \mu_{ij}, \qquad \mu_{ij} = -\langle M_i{}^* v_j \rangle_j \tag{1.7}$$

2. Changes in the moment of inertia. Let us multiply the continuity equation valid [1] in the microscale $dx_1 dx_2 dx_3$ of a turbulized fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} \doteq 0$$

by $\xi_k \xi_m$. Then, taking the average with respect to the vloume V, we obtain

$$\frac{\partial}{\partial t} \langle \rho \xi_k \xi_m \rangle + \frac{\partial}{\partial X_j} \langle \rho \xi_k \xi_m u_j \rangle_j = \langle \rho u_j \xi_m \delta_{kj} \rangle + \langle \rho u_j \xi_k \delta_{mj} \rangle \tag{2.1}$$

It is easy to perform the following transformations:

$$\langle \rho \xi_k \xi_m \rangle = I_{km} + \langle i_{km} \rangle$$

as well as to determine the pulsation $i_{km}{}^*$ of the moment of inertia

$$\rho \xi_k \xi_m = \langle \rho \xi_k \xi_m \rangle + i_{km}{}^*$$

Then the flux of the moment of inertia is determined as follows:

$$\langle \rho \xi_k \xi_m u_j \rangle_j = \langle \rho \xi_k \xi_m \rangle U_j + \langle i_{km}{}^* v_j \rangle_j$$

and Eq. (2.1) takes the form

$$\frac{\partial}{\partial t} \langle \rho \xi_k \xi_m \rangle + \frac{\partial}{\partial X_j} \langle \rho \xi_k \xi_m \rangle U_j = \frac{\partial U_k}{\partial X_n} \langle \rho \xi_n \xi_m \rangle + \frac{\partial U_m}{\partial X_n} \langle \rho \xi_n \xi_k \rangle + \left\langle \frac{\partial w_k}{\partial \zeta_n} i_{mn} + \frac{\partial w_m}{\partial \zeta_n} i_{nk} \right\rangle - \frac{\partial}{\partial X_j} \langle i_{km}{}^* v_j \rangle_j \tag{2.2}$$

The equation of the changes in the moment of inertia (2.2) extends the equation derived earlier in [6] for a fluid with inner structure and in [5] for an ordinary fluid, to the case of turbulent flux. The intrinsic moment of inertia of liquid-crystalline media was considered in [7]. If it is considered that the moment of inertia I_{km} of a fluid in a unit volume V sarisfies the work equation [5]

$$\frac{\partial I_{km}}{\partial t} + U_j \frac{\partial I_{km}}{\partial X_j} = \frac{\partial U_k}{\partial X_n} I_{nm} + \frac{\partial U_m}{\partial X_n} I_{nk} \tag{2.3}$$

then by forming the difference between (2.2) and (2.3), we find

$$\frac{\partial}{\partial t} (J \delta_{km}) + \frac{\partial}{\partial X_j} (J \delta_{km} U_j) = 2 \langle i e_{km} \rangle - \frac{\partial}{\partial X_j} \langle i^* v_j \rangle_j \tag{2.4}$$

where e_{ik} is the strain rate tensor (see formulas (3.6)), and

$$i_{km} = {}^1\!/_3 i \delta_{km}, \qquad \langle i_{km} \rangle = {}^1\!/_3 J \delta_{km}$$

If changes in the moment of inertia J are neglected because of local strains, then (2.4) becomes

$$\frac{\partial J}{\partial t} + U_j \frac{\partial J}{\partial X_j} = -\frac{\partial}{\partial X_j} \langle i^* v_j \rangle_j \tag{2.5}$$

The modification of the theory in which

$$\frac{\partial J}{\partial t} + U_j \frac{\partial J}{\partial X_j} = 0, \qquad \frac{\partial}{\partial X_j} \langle i^* v_j \rangle_j = 0$$

is considered in [1] and herein later.

3. **Total energy balance.** Let us write the total energy balance of a fluid in integral form for an arbitrary volume V_1 fixed in space and bounded by a surface S_1:

$$\frac{\partial}{\partial t}\int_{V_1}\rho\left(e+\frac{u_iu_i}{2}\right)dV+\int_{S_1}\rho\left(e+\frac{u_iu_i}{2}\right)u_n\,dS_n=\int_{S_1}t_{in}u_i\,dS_n+\int_{S_1}q_n\,dS_n+$$
$$\int_{V_1}Q\,dV+\int_{S_1}c_{in}\Phi_i\,dS_n+\int_{V_1}F_ku_k\,dV+\int_{V_1}\rho G_k\Phi_k\,dV \tag{3.1}$$

where e is the inner energy of the fluid, q is the heat flux, Q are the internal heat sources, Φ_i is the vector of the total angular velocity of a fluid particle. Henceforth, as in (1.1), we shall assume that the moment stresses C_{in} and the volume moments G_k governed by the molecular structure of the fluid are identically zero.

If we select the volume $V=\Delta X_1\Delta X_2\Delta X_3\sim\Delta^3$ as the volume V_1, where Δ is the lonear scale of V, then (3.1) can be represented as

$$\frac{\partial\langle\rho e\rangle}{\partial t}+\frac{\partial}{\partial X_j}\langle\rho eu_j\rangle+\frac{\partial}{\partial t}\left\langle\frac{1}{2}u_iu_i\right\rangle+\frac{\partial}{\partial X_j}\left\langle\frac{1}{2}\rho u_iu_iu_j\right\rangle_j=$$
$$\frac{\partial}{\partial X_j}\langle t_{ij}u_i\rangle_j+\frac{\partial\langle q_j\rangle_j}{\partial X_j}+\langle Q\rangle+\langle F_ku_k\rangle \tag{3.2}$$

Let us take the average of the kinetic energy of the turbulized fluid by using the representation (1.3)

$$\frac{1}{2}\rho u_iu_i=\frac{1}{2}\rho U_iU_i+\rho U_i\left(\frac{\partial U_i}{\partial X_k}\xi_k+v_i\right)+\frac{1}{2}\rho w_iw_i+$$
$$\frac{1}{2}\rho\frac{\partial U_i}{\partial X_k}\frac{\partial U_i}{\partial X_m}\xi_k\xi_m+\frac{1}{2}\rho\frac{\partial w_i}{\partial\zeta_k}\frac{\partial w_i}{\partial\zeta_n}(\xi_k-\zeta_k)(\xi_n-\zeta_n)+$$
$$\frac{\partial U_i}{\partial X_k}\frac{\partial w_i}{\partial\zeta_m}\xi_k(\xi_m-\zeta_m)+\rho w_i\frac{\partial U_i}{\partial X_k}\xi_k+\rho w_i\frac{\partial w_i}{\partial\zeta_k}(\xi_k-\zeta_k) \tag{3.3}$$

We take the average of (3.3) over the volume ΔV

$$\frac{1}{2}\overline{\rho u_iu_i}=\frac{1}{2}\bar\rho U_iU_i+\bar\rho U_i\left(\frac{\partial U_i}{\partial X_k}\zeta_k+w_i\right)+\frac{1}{2}\bar\rho w_iw_i+$$
$$\frac{1}{2}\rho\frac{\partial U_i}{\partial X_k}\frac{\partial U_i}{\partial X_k}\zeta_k\zeta_k+\left(\frac{\partial U_i}{\partial X_m}+\frac{\partial w_i}{\partial\zeta_m}\right)\left(\frac{\partial U_i}{\partial X_k}+\frac{\partial w_i}{\partial\zeta_m}\right)i_{km} \tag{3.4}$$

Subsequent averaging over all the volumes ΔV contained in V yields

$$\left\langle\frac{1}{2}\rho u_iu_i\right\rangle=\frac{1}{2}\langle\rho\rangle U_iU_i+\frac{1}{2}I_{km}\frac{\partial U_i}{\partial X_k}\frac{\partial U_i}{\partial X_m}+\frac{1}{2}\langle\rho w_iw_i\rangle+$$
$$\frac{1}{2}\left\langle i_{km}\left(\frac{\partial U_i}{\partial X_m}+\frac{\partial w_i}{\partial\zeta_m}\right)\left(\frac{\partial U_i}{\partial X_k}+\frac{\partial w_i}{\partial\zeta_k}\right)\right\rangle \tag{3.5}$$

It can be shown that the last summand in (3.5) is represented as

$$\frac{1}{2}\left\langle i_{km}\left(\frac{\partial U_i}{\partial X_m}+\frac{\partial w_i}{\partial\zeta_m}\right)\left(\frac{\partial U_i}{\partial X_k}+\frac{\partial w_i}{\partial\zeta_k}\right)\right\rangle=\frac{1}{4}\langle ie_{ik}e_{ik}\rangle+\frac{1}{2}\langle i\Phi_j\Phi_j\rangle$$
$$e_{ik}=\frac{1}{2}\left(\frac{\partial U_i}{\partial X_k}+\frac{\partial U_k}{\partial X_i}\right)+\frac{1}{2}\left(\frac{\partial w_i}{\partial\zeta_k}+\frac{\partial w_k}{\partial\zeta_i}\right) \tag{3.6}$$

Therefore, a contribution to the kinetic energy introduced by the strain rate field e_{ik} in the scale d appears. Neglecting this effect, as well as the energy generated by the mean gradient in the scale Δ, we obtain

$$\langle {}^1/_2 \rho u_i u_i \rangle = {}^1/_2 \langle \rho \rangle U_i U_i + {}^1/_2 \langle \rho w_i w_i \rangle + {}^1/_2 \langle i \Phi_i \Phi_i \rangle \tag{3.7}$$

We furthermore transform the last summand in (3.6)

$$ {}^1/_2 \langle i \Phi_j \Phi_j \rangle = {}^1/_2 \langle (J + i^*)(\Omega_j + \Phi_j^*)(\Omega_j + \Phi_j^*) \rangle = $$
$$ {}^1/_2 J (\Omega_j + \omega_j)(\Omega_j + \omega_j) + {}^1/_2 \langle i \Phi_j^* \Phi_j^* \rangle - {}^1/_2 J \omega_j \omega_j $$

Inserting this result into (3.7), we find

$$\langle {}^1/_2 \rho u_i u_i \rangle = {}^1/_2 \langle \rho \rangle U_i U_i + {}^1/_2 J (\Omega_j + \omega_j)(\Omega_j + \omega_j) + \langle \rho \rangle E \tag{3.8}$$

where E is the inner energy of the turbulent field

$$E = E_w + E_\omega \tag{3.9}$$

$$\langle \rho \rangle E_w = \langle {}^1/_2 \rho w_i w_i \rangle, \qquad \langle \rho \rangle E_\omega = {}^1/_2 \langle i \Phi_j^* \Phi_j^* \rangle - {}^1/_2 J \omega_j \omega_j$$

We define the kinetic energy pulsation as follows:

$$({}^1/_2 \rho u_i u_i)^* = U_i (\rho v_i) + (\Omega_i + \omega_i) M_i^* + (\rho E)^* \tag{3.10}$$

where $E^* = E_w^* + E_\omega^*$ is selected so as to comply with the condition

$$\langle {}^1/_2 \rho u_i u_i \rangle + {}^1/_2 (\rho u_i u_i)^* = {}^1/_2 \rho u_i u_i \tag{3.11}$$

Let us transform the kinetic energy flux analogously to the momentum and moment of momentum fluxes

$$\left\langle \frac{1}{2} \rho u_i u_i u_j \right\rangle_j = \left\langle \left[\left\langle \frac{1}{2} \rho u_i u_i \right\rangle + \left(\frac{1}{2} \rho u_i u_i \right)^* \right] \left(U_i + \frac{\partial U_j}{\partial X_k} \xi_k + v_j \right) \right\rangle_j$$

Hence

$$\langle {}^1/_2 \rho u_i u_i u_j \rangle_j \approx \langle {}^1/_2 \rho u_i u_i \rangle U_j + \langle ({}^1/_2 \rho u_i u_i)^* v_j \rangle_j$$

We now use the representation (3.10)

$$\langle (\frac{1}{2} \rho u_i u_i)^* v_j \rangle_j = U_i \langle \rho v_i v_j \rangle_j + (\Omega_i + \omega_i) \langle M_i^* v_j \rangle_j + \langle (\rho E)^* v_j \rangle_j$$

and finally obtain

$$\langle \frac{1}{2} \rho u_i u_i u_j \rangle_j = \langle \frac{1}{2} \rho u_i u_i \rangle U_j + (\rho E)^* v_j \rangle_j - U_i R_{ij} - (\Omega_i + \omega_i) \mu_{ij} \tag{3.12}$$

Let us note that it is also possible to carry out the following transformations:

$$\langle \rho e u_j \rangle = \langle \rho \rangle \langle e \rangle U_j + \langle (\rho e)^* v_j \rangle_j, \qquad \varepsilon_{ilk} \langle \rho u_l u_k \rangle_k = \varepsilon_{ilk} R_{lk}$$

$$\langle t_{ij} u_i \rangle_j = \tau_{ij} U_i + \langle t_{ij}^* v_i \rangle_j, \qquad \tau_{ij} = \langle t_{ij} \rangle_j, \qquad \langle \varepsilon_{ilk} \xi_l F_k \rangle = C_i$$

$$\langle F_k u_k \rangle \approx \langle F_k \rangle U_k + \langle F_k^* v_k \rangle, \qquad \langle F_k^* v_k \rangle = C_i (\Omega_i + \omega_i) + \Pi \tag{3.13}$$

4. Motion and energy equations. Now, we substitute the resultant averaged expressions into the balance equations of the mass, momentum, moment of momentum and energy. We then obtain [1] the differential equations of motion in the following form:

$$\frac{\partial \langle\rho\rangle}{\partial t} + \frac{\partial \langle\rho\rangle U_j}{\partial X_j} = 0 \tag{4.1}$$

$$\frac{\partial \langle\rho\rangle U_i}{\partial t} + \frac{\partial \langle\rho\rangle U_i U_j}{\partial X_j} = \frac{\partial R_{ij}}{\partial X_j} + \frac{\partial \tau_{ij}}{\partial X_j} + \langle F_i\rangle \tag{4.2}$$

$$\frac{\partial}{\partial t} J(\Omega_i + \omega_i) + \frac{\partial}{\partial X_j} J(\Omega_i + \omega_i) U_j = \frac{\partial \mu_{ij}}{\partial X_j} + \varepsilon_{ilk}(R_{lk} + \tau_{lk}) + C_i \tag{4.3}$$

$$\frac{\partial J}{\partial t} + \frac{\partial J U_j}{\partial X_j} = -\frac{\partial}{\partial X_j}\langle i^* v_j\rangle_j = 0 \tag{4.4}$$

The total energy balance equation (3.1) becomes in conformity with (3.8), (3.12), (3.13)

$$\langle\rho\rangle\left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial X_j}\right)\left(e + \frac{1}{2}U_iU_i + \frac{1}{2}\frac{J}{\langle\rho\rangle}(\Omega_i + \omega_i)(\Omega_i + \omega_i) + E\right) =$$
$$\frac{\partial}{\partial X_j}(U_iR_{ij} + (\Omega_i + \omega_i)\mu_{ij} + \tau_{ij}U_i) + Q + \Pi + \langle F_k\rangle U_k + C_k(\Omega_k + \omega_k) +$$
$$\frac{\partial}{\partial X_j}(-\langle(\rho e)^* v_j\rangle_j - \langle(\rho E)^* v_j\rangle_j + \langle t_{ij}{}^* v_i\rangle_j + \langle q_j\rangle_j) \tag{4.5}$$

Now multiplying Eq. (4.1) by U_i and Eq. (4.2) by the total angular velocity $\Omega_i + \omega_i$, we find the equations for the kinetic energies of the mean translational motion and the mean rotational motion

$$\langle\rho\rangle\left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial X_j}\right)\left(\frac{1}{2}U_iU_i\right) = U_i\frac{\partial R_{ij}}{\partial X_j} + U_i\frac{\partial \tau_{ij}}{\partial X_j} + U_i\langle F_i\rangle \tag{4.6}$$

$$J\left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial X_j}\right)\left(\frac{1}{2}(\Omega_i + \omega_i)(\Omega_i + \omega_i)\right) = (\Omega_i + \omega_i)\frac{\partial \mu_{ij}}{\partial X_j} +$$
$$\varepsilon_{ilk}(R_{lk} + \tau_{lk})(\Omega_i + \omega_i) + C_i(\Omega_i + \omega_i) \tag{4.7}$$

Subtracting Eqs. (4.6), (4.7) from the total energy equation (4.5) and taking assount of (4.1) and (4.4), we obtain the heat influx equation (the equation for the total inner energy of the turbulized fluid) (4.8)

$$\rho\left(\frac{\partial}{\partial t} + U_i\frac{\partial}{\partial X_j}\right)(e + E) = (R_{ij}{}^s + \tau_{ij}{}^s)\frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) + \mu_{ij}\frac{\partial(\Omega_i + \omega_i)}{\partial X_j} =$$
$$-(R_{ij}{}^a + \tau_{ij}{}^a)\varepsilon_{ijk}\omega_k + \frac{\partial}{\partial X_j}(-\langle(\rho e)^* v_j\rangle_j - \langle(\rho E)^* v_j\rangle_j + \langle t_{ij}v_j\rangle_j + \langle q_j\rangle_j)$$

Here R_{ij}^s, R_{ij}^a, τ_{ij}^s, τ_{ij}^a are, respectively, the symmetric and antisymmetric components of the stress tensors R_{ij}, τ_{ij}, for example

$$R_{ij}^s = {}^1/_2(R_{ij} + R_{ji}), \qquad R_{ij}^a = {}^1/_2(R_{ij} - R_{ji})$$

The derivation of (4.8) differs from the ordinary derivation [8] of the turbulent energy equation by taking account of the presence of the rotational degrees of freedom of the system characteristic for asymmetric hydromechanics [9].

5. Entropy of a turbulized fluid. The balance equation of the entropy s in the microvolume dV is

$$\frac{\partial \rho s}{\partial t} + \frac{\partial(\rho s u_i)}{\partial x_i} = \sigma + \frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) \tag{5.1}$$

where q_i / T is the entropy flux, T is the temperature, and σ is the rate of local entropy production
$$\sigma = \frac{1}{T}(t_{ij} + p\delta_{ij})\frac{\partial u_i}{\partial x_j} + \frac{q_i}{T^2}\frac{\partial T}{\partial x_i} + \frac{Q}{T}$$
The entropy balance of a turbulent system is formed [10] by a statistical averaging of Eq.(5.1). Herein, as in [11], the procedure for taking the average of (5.1) is performed over the volume V but the necessity of taking the average over the areas [1] also is hence taken into account
$$\langle\rho\rangle\left(\frac{\partial\langle s\rangle}{\partial t} + U_i\frac{\partial\langle s\rangle}{\partial X_i}\right) = \langle\sigma\rangle + \frac{\partial}{\partial X_i}\left(\frac{\langle q_i\rangle_i}{\langle T\rangle} - \langle(\rho s)^* v_i\rangle_i\right) \tag{5.2}$$
$$\langle\sigma\rangle = \frac{1}{\langle T\rangle}(\tau_{ij} + \langle p\rangle\delta_{ij})\left(\frac{\partial U_i}{\partial X_j}\right) + \frac{1}{\langle T\rangle}\varphi + \frac{1}{\langle T\rangle}\langle Q\rangle + \frac{\langle q_i\rangle_i}{\langle T\rangle^2}\frac{\partial\langle T\rangle}{\partial X_j}$$
$$\varphi = \left\langle(t_{ij} - \tau_{ij})\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial U_i}{\partial X_j}\right)\right\rangle = \left\langle t_{ij}^*\frac{\partial v_i}{\partial x_j}\right\rangle$$
For simplicity, the effects of temperature and pressure pulsations (cf [11]) are omitted here.

Therefore, the production of the averaged entropy s corresponding to the transition of mechanical energy into heat is determined by the work of the mean viscous stresses τ_{ij} over the field of mean velocities U_j as well as the inner source φ which corresponds to the additional work of viscous stresses due to turbulization of the fluid. Moreover, the heat fluxes, governed by the influence of turbulization of the contact heat conduction $\langle q_i\rangle_i$ appear in (5.2), and the heat transfer $\langle(\rho s)^* v_j\rangle_j$ which is convective in nature in the scale dV but turbulent diffusion in the scale V, is taken into account also.

We emphasize that together the first two summands in the expression for $\langle\sigma\rangle$ correspond to the total viscous dissipation of mechanical energy into heat (in the case of no transverse shear the viscous dissipation reduces to an energy sink φ). If the balance equation of the inner energy (heat influx) which is valid in the volume dV
$$\frac{\partial\rho e}{\partial t} + \frac{\partial(\rho e u_i)}{\partial x_i} = t_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} + Q \tag{5.3}$$
is averaged in an analogous manner over the volume V, then we obtain
$$\langle\rho\rangle\left(\frac{\partial\langle e\rangle}{\partial t} + U_i\frac{\partial\langle e\rangle}{\partial X_i}\right) = \tau_{ij}\frac{\partial U_i}{\partial X_j} + \varphi + \frac{\partial\langle q_i\rangle}{\partial X_i} + \langle Q\rangle + \frac{\partial}{\partial X_i}\langle-(\rho e)^* v_i\rangle_i \tag{5.4}$$
The Gibbs relationship for the mean entropy $\langle s\rangle$ and energy $\langle e\rangle$
$$\langle\rho\rangle\frac{d\langle e\rangle}{dt} = \langle T\rangle\frac{d\langle s\rangle}{dt} - p\frac{d}{dt}\frac{1}{\langle\rho\rangle}, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + U_j\frac{\partial}{\partial X_j} \tag{5.5}$$
follows from a comparison of (5.2) and (5.4) if we use the equality (cf [10])
$$\frac{\partial}{\partial X_i}\langle(\rho e)^* v_i\rangle_i + \langle T\rangle\frac{\partial}{\partial X_i}\langle(\rho s)^* v_i\rangle_i = 0$$
Let us subtract (5.4) from the equation of the total inner energy of a turbulent field. We then obtain the equation determining the inner energy of the intrinsically turbulent superstructure
$$\rho\left(\frac{\partial E}{\partial t} + U_j\frac{\partial E}{\partial X_j}\right) = R_{ij}^s\frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) -$$
$$R_{ij}^a\varepsilon_{ijk}\omega_k + \mu_{ij}\frac{\partial(\Omega_i + \omega_i)}{\partial X_j} + \frac{\partial}{\partial X_j}\langle-\rho E^* v_j\rangle_j + \Psi \tag{5.6}$$

where the turbulent energy sink is

$$\Psi = \frac{\partial \langle t_{ij}^{*} v_i \rangle_j}{\partial X_j} - \varphi \tag{5.7}$$

We note that the viscous dissipation due to pulsations is interpreted as an internal sink in the turbulent energy equation in [12].

The work of the turbulent stresses R_{ij}, μ_{ij} over the field of mean translational and angular velocities results in dissipation of the mean field mechanical energy into the energy of chaotic turbulent motion, which is "thermal" in nature in the scale V (but mechanical in the scale dV). The turbulent entropy S and turbulization temperature Θ can be introduced correspondingly as follows:

$$\Theta \frac{dS}{dt} = \left(R_{ij}^{s} - \frac{1}{3} R_{kl}\delta_{kl}\delta_{ij}\right)\frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) - R_{ij}^{a}\varepsilon_{ijk}\omega_k + \mu_{ij}\frac{\partial(\Omega_i + \omega_i)}{\partial X_j} + \frac{\partial}{\partial X}\langle -\rho E^{*} v_j\rangle_j + \Psi$$

or

$$\frac{dS}{dt} = \Sigma + \frac{\partial}{\partial X_j}\left(\frac{\langle -\rho E^{*} v_j\rangle_j}{\Theta}\right) + \frac{\Psi}{\Theta} \tag{5.8}$$

Here Σ is the local generation of turbulent entropy, and the quantity Ψ / Θ is a sink of the turbilent entropy S. Thus, the work of the Reynolds and other turbulent stresses results in growth of the entropy (chaos) of turbulence, and viscous dissipation diminishes the entropy (chaos) of turbulation.

Let us examine the particular case of a local stationary state. We neglect turbulent-diffusion energy transfer and the pulsation work of viscous stresses on the boundaries of the volume V. Then the sink is $\Psi = -\varphi$, and Eq. (5.8) of the growth in turbulization entropy reduces to the following:

$$dS/dt = \Sigma - \varphi/\Theta = 0 \tag{5.9}$$

Thus, in the stationary case the positive production Σ of turbulent entropy S must be compensated by the negative influx of entropy Ψ / Θ (or the positive influx of negentropy). In other words, in a specific sense a turbulent field is similar to a biological system [3, 4]. Indeed, the internal configurations of both systems are sustained because of the continuous influx of energy. It should be emphasized that this remark on the influx of negative entropy corresponds substantially to the known Richardson-Kolmogorov principle about the energy balance of the equilibrium hierarchy of vortices [8]. Furthermore, from (5.9) in this stationary case we have that the work of the turbulent stresses equals the volume viscous dissipation, and in combination with the work of the mean viscous stresses yields the total dissipation of the mechanical energy into heat.

The Gibbs relationship for a turbulent field

$$\frac{dE}{dt} = \Theta\frac{dS}{dt} + \frac{1}{3} R_{ij}\delta_{ij}\frac{d}{dt}\frac{1}{\langle\rho\rangle} \tag{5.10}$$

follows from (5.6) and (5.8), i.e. the parameters of thr state of the turbulent system are the temperature Θ and the mean density $\langle\rho\rangle$.

According to (3.9), the inner energy E consists of two parts, the translational E_v and the rotational E_ω (the inner energy of turbulization), hence, a rather more general construction can be carried out by introducing the two entropies S_v and S_ω and the two temperatures θ_v and θ_ω, respectively:

$$\frac{dE_w}{dt} + \Lambda_{ij}{}^w \frac{d\chi_{ij}{}^w}{dt} = \theta_w \frac{dS_w}{dt} + \frac{1}{3}(R_{ij}\delta_{ij}) \frac{d}{dt}\frac{1}{\langle\rho\rangle} \tag{5.11}$$
$$\frac{dE_\omega}{dt} + \Lambda_{ij}{}^\omega \frac{d\chi^\omega}{dt} = \theta_\omega \frac{dS_\omega}{dt}$$

where Λ_{ij}^w and Λ_{ij}^ω are some forces (stresses) working on the displacements (strain increments) $d\chi_{ij}^w$ and $d\chi_{ij}^\omega$. Together the relations (5.11) yield

$$\frac{dE}{dt} = \theta\frac{dS}{dt} + (\theta_\omega - \theta)\frac{dS_\omega}{dt} + \Lambda_{ij}{}^\omega \frac{d\chi_{ij}{}^\omega}{dt} + \Lambda_{ij}{}^\omega \frac{d\chi_{ij}{}^\omega}{dt} + \frac{1}{3}(R_{ij}\delta_{ij})\frac{d}{dt}\frac{1}{\langle\rho\rangle} \tag{5.12}$$

where $\theta = \theta_w$, $dS = dS_w + dS_\omega$ is the increment in the total entropy of turbulence. Now, if the inner energy of the turbulent field is eliminated from (5.6) and (5.12), we then obtain a generalized equation for the entropy balance

$$\theta\langle\rho\rangle\frac{dS}{dt} = (\theta - \theta_\omega)\langle\rho\rangle\frac{dS_\omega}{dt} + (R_{ij}{}^s - r_{ij}{}^s)\left(\frac{\partial U_i}{\partial X_j}\right)^s - (R_{ij}{}^a - r_{ij}{}^a)\,\varepsilon_{ijk}\omega_k +$$
$$(\mu_{ij} - \eta_{ij})\left(\frac{\partial\Omega_0}{\partial X_j} + \frac{\partial\omega_0}{\partial X_j}\right) + \frac{\partial}{\partial X_j}\langle -\rho E^* v_j\rangle_j + \frac{\partial}{\partial X_j}\langle\ldots\rangle_j + \Psi \tag{5.13}$$

Here $r_{ij}{}^s$, r_{ij}^a, η_{ij} are some stress tensor components; they can be expressed in terms of Λ_{ij}^w, Λ_{ij}^ω, if $d\chi_{ij}^w/dt$, $d\chi_{ij}^\omega/dt$ are related linearly to the tensors $\partial U_i/\partial X_j$ and $\partial(\Omega_i + \omega_i)/\partial X_j$. The interoduction of these quantities correspond to taking account of the "elastic" properties of the turbulization. The possibility of such effects is mentioned in [13 – 15]. It is absolutely necessary to take them into account in analyzing the turbulization of non-Newtonian fluids.

The inequality of the temperatures $\theta \neq \theta_\omega$ permits obtaining the nonequilibrium transition of turbulent field energy from translational to rotational "degrees of freedom"(*).

6. The closure problem. To find the governing relations (between the dynamic and kinematic variables), we can use the Onsager formalism of the thermodynamics of irreversible processes. Let us say that the introduction of turbulent viscosity is essentially a particular case of such an approach.

If we proceed from the requirement of a local growth in the total entropy $dS_t = dS + ds$, then the mutual influence of the strains generating the entropy in the microscale (dV) and the macroscale (V) will consequently be taken into account. However, we will consider that turbulization exerts no influence on the relation between τ_{ij} and $\partial U_i/\partial X_i$, say. Correspondingly, we will use the requirements of positivity of $\langle\sigma\rangle$ and Σ independently. The difficulties hence are related to the constructions for the streams in the scale V, i.e. for the turbulent field.

The closing relationship between thermodynamic streams (of momentum and moment of momentum) J_α and the characteristics of the averaged field X_β can be derived with the use of the generalized [17] Onsager principle

$$J_\alpha = \int dt' \int L_{\alpha\beta}(x - x', t - t')\, X_\beta(x', t')\, dx' \tag{6.1}$$

where $L_{\alpha\beta}$ is the matrix of Onsager coefficients of functions of the turbulent field structure. This relationship is nonlocal, and yields averaged relationships for a rapidly

*) The thermodynamical analysis of the hierarchy of Richardson-Kolmogorov vortices naturally requires the introduction of a whole spectrum of "degrees of freedom", and the energy, temperature, and entropy, respectively, see [16].

changing field structure. Formulas of the kind of (6.1) may be considered suitable for calculating turbulent flows in alternating modes [8]. Equations (6.1) can usually be reduced to

$$J_\alpha = L_{\alpha\beta} X_\beta \tag{6.2}$$

where $L_{\alpha\beta}$ are functions of the point at which J_α and X_β are defined and summation is carried out with respect to recurrent subscripts. Nonzero elements of $L_{\alpha\beta}$ and $\alpha \neq \beta$ make it possible to take into account the cross effects and the Curie rule to separate the interaction between streams of even and odd tensor dimension. New cross effects may obviously appear in asymmetric mechanics. For instance, the presence of a peculiar thermomechanical effect related to the asymmetry of the moment-stress tensor was pointed out in [9].

Since the structure of the turbulent field depends itself on streams J_α, the related Onsager type formulas are complex nonlinear relationships (while the effectiveness of linear formulas (6.2) is to a considerable extent lost). Owing to this, the matrix of coefficients $L_{\alpha\beta}$ in the case of a turbulent field depends not only on parameters of state (e.g. on turbulence temperature ϑ) but, also, on the averaged parameters of velocity field X_β (i.e. on tensors $\partial U_i / \partial X_j$, $\varepsilon_{ijk}\omega_k$, $\partial\langle e'v_j\rangle_j / \partial X_j$, ...). The interaction of various streams X_β can affect $L_{\alpha\beta}$, which may result in additional cross effects. The dependence of internal structure on streams X_β implies that the exclusion of one (or a part of it) of the X_β streams may result not only in the change of cross coefficients $L_{\alpha\beta}$ but, also, of diagonal elements of that matrix. Because of this the conventional requirement for positive determinacy of each of the terms in the sum of products

$$\sum = L_{\alpha\beta} X_\beta J_\alpha$$

used in [11] is no longer applicable, and the only requirement is that Σ must be strictly positive. Owing to this, the superposition of various streams can theoretically yield negative individual elements of matrix $L_{\alpha\beta}$. This can possibly explain the effect of the so-called negative viscosity [18]. However, it should be borne in mind that the inference of the existence in certain flows (see [19]) of negative turbulent viscosity is based on the comparison of profiles of averaged values of Ω_i and R_{ij} (or U and R_{ij} in the problem of flow in a circular channel [19, 20]) and their conventional interpretation without taking into consideration the antisymmetric component of Reynolds stresses. Whether it is sufficient to allow in such cases for asymmetric effects (see [20]) or it will be necessary to introduce negative transport coefficients, can only be determined by the comparison of specific calculations with experimental data.

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Translated by M.D.F.